Bachelor of Science (B.Sc.) Semester—III (C.B.S.) Examination

MATHEMATICS

(M_5-Advanced Calculus, Sequence and Series)

Paper—I

Time : Three Hours] [Maximum Marks : 60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question Nos. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) If a function f(x) is continuous on [a, b] and derivable in (a, b) then prove that there exists a point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$ 6

(B) Show that :

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2} \; ; \; 0 < u < v, \text{ and deduce that}$$

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}.$$ 6

OR

(C) Let f(x, y) and g(x, y) be defined in the open region $D \subset R^2$. If f(x, y) and g(x, y) both are continuous at $P_o(x_o, y_o) \in D$, then prove that $f(x, y) \cdot g(x, y)$ is also continuous at $P_o(x_o, y_o)$. 6

(D) Expand $f(x, y) = x^2 + xy + y^2$ in the powers of $(x - 2)$ and $(y - 3)$ by Taylor's theorem. 6

UNIT—II

2. (A) Find the envelope of the family of lines $x \cos^3 \alpha + y \sin^3 \alpha = c$, where α being a parameter and c is a constant. 6

(B) Find the envelope of the family of parabolas $(x/a)^{1/2} + (y/b)^{1/2} = 1$ when $a^n + b^n = c^n$, where a and b are the parameters and c being a constant. 6

OR

(C) Show that the minimum value of

$$x = xy + \left(\frac{a^3}{x}\right) + \left(\frac{a^3}{y}\right) \text{ is } 3a^2.$$ 6

(D) Find the minimum value of $x^2 + y^2 + z^2$ when $yz + zx + xy = 3a^2$ by using Lagrange's Multiplier Method. 6

UNIT—III

3. (A) Let $< x_n >$ and $< y_n >$ be two sequences such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$.

Then prove that :

$$\lim_{n \to \infty} [x_n + y_n] = x + y.$$ 6

(B) Show that the sequence $< x_n >$ defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ is monotonic increasing and converges to 2. 6

OR

(C) Prove that every Cauchy Sequence is bounded. Give an example to show that the converse is not true. 6

(D) By applying Cauchy's convergence criterion show that the sequence $< x_n >$ given by

$$x_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots.. + \frac{1}{2n-1} + \ldots. \text{ diverges.}$$ 6

UNIT—IV

4. (A) Test the convergence of the series

(i) $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$

(ii) $\sum_{n=1}^{\infty} \frac{n+1}{(n+2)^2}$,

by comparison test. 6

(B) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $0 < p \le 1$ by using integral test. 6

OR

(C) Show that the series

$$\frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \ldots.. \text{ is}$$

conditionally convergent. 6

(D) Test the convergence of the series

$$2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \ldots. \frac{(n+1)x^n}{n^3} + \ldots.$$

by ratio test. 6

Question—5

5. (A) Verify Rolle's theorem for

$f(x) = 2 + (x + 1)^{2/3}$ for $x \in [0, 2]$. 1½

(B) Using ε–δ definition, show that

$$\lim_{(x,y)\to(1,1)} (x+2y) = 3.$$ 1½

(C) If A, B, C are the functions of x and y and m is a parameter then show that the envelope of $Am^2 + Bm + C = 0$ is $B^2 = 4\,AC$. 1½

(D) Find the critical points of

$$u = x^3 + y^2 - 3axy.$$ 1½

(E) Find $n_o \in N$ such that

$$\left|\frac{n}{n+3} - 1\right| < \frac{1}{5} \text{ for all } n \geq n_o.$$ 1½

(F) Show that the sequence $\left\langle \frac{n}{n+1} \right\rangle$, $\forall$ $n \in N$ is bounded. 1½

(G) Test the convergence of the series $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$. 1½

(H) Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{x \log x}$ by integral test. 1½